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Base subsets of Grassmannians: Infinite-dimensional case

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Abstract

Let V be an infinite-dimensional vector space. We define Grassmannians of V as orbits of the action of the group $\mathrm{GL}(V)$ on the set of proper subspaces of V and study transformations of Grassmannians preserving so-called base subsets.

MSC: 51M35; 14M15

1. Introduction

Let V be a vector space over a division ring and $\mathcal{S}$ be the set of proper subspaces of V. The group $\mathrm{GL}(V)$ acts on $\mathcal{S}$. The orbits of this action are said to be *Grassmannians*. If $\dim V < \infty$ then each Grassmannian is

$$\mathcal{G}_k := \{S \in \mathcal{S} \mid \dim S = k\},$$

where $k \in \{1, \ldots, \dim V - 1\}$. The infinite-dimensional case is more complicated.

Two elements S and U of a Grassmannian are called *adjacent* if

$$\dim(S/(S \cap U)) = \dim(U/(S \cap U)) = 1.$$

By this definition, any two distinct 1-dimensional subspaces are adjacent and the same holds for hyperplanes. The *Grassmann graph* of a Grassmannian $\mathcal{G}$ is the graph whose vertexes are elements of $\mathcal{G}$ and whose edges are pairs of adjacent vertexes.

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If $\dim V = n < \infty$ then this graph is connected and the well-known Chow's theorem [6] says that any automorphism of the Grassmann graph of $\mathcal{G}_k$ $(1 < k < n-1)$ is induced by a semilinear isomorphism of V to itself or to the dual vector space V^* (the second possibility can be realized only for the case when $n = 2k$). Some results connected to Chow's theorem can be found in [4,8–10,16]. We also refer to [7,11,12] for combinatorial characterizations of finite Grassmann graphs.

It was pointed out by Blunck and Havlicek [3] that if $\dim V = \infty$ and elements of a Grassmannian of V have infinite dimension and codimension then the associated Grassmann graph is not connected and there exist automorphisms which are not induced by semilinear isomorphisms (the restrictions of such automorphisms to different connected components of the Grassmann graph are induced by different elements of $\mathrm{P\Gamma L}(V)$, see Example 4.3 in [3]).

Let $\mathcal{G}$ be a Grassmannian. Each base of V defines the *base subset* of $\mathcal{G}$ consisting of all elements of $\mathcal{G}$ spanned by vectors of this base. Let us consider the block space [5] whose points are elements of $\mathcal{G}$ and whose blocks are the base subsets of $\mathcal{G}$.

If $3 \le \dim V = n < \infty$ then each automorphism of the block space associated with $\mathcal{G}_k$ is induced by a semilinear isomorphism of V to itself or to V^* (the second possibility can be realized only for $n = 2k$). If $k = 1, n-1$ then this is a simple consequence of the Fundamental Theorem of Projective Geometry. For the case when $1 < k < n-1$ this statement was proved by the author [13]; he has shown that automorphisms of the block space defined above are automorphisms of the Grassmann graph; more general results were given in the author's later papers [14,15].

Base subsets can be considered as shadows of apartments of the building over V [17]. It follows from the results of Abramenko and Van Maldeghem [1] that apartment preserving transformations of the chamber set of a spherical building are induced by automorphisms of the associated chamber complex.

In the present paper we give an infinite-dimensional analogue of [13]. Since Chow's theorem does not hold for the Grassmann graphs of infinite-dimensional vector spaces, the methods of [13–15] cannot be applied to this case.

2. Result

From this moment we suppose that $\dim V = \aleph$ is infinite. Then for any subspace $S \subset V$ we have

$$\aleph = \max\{\dim S, \operatorname{codim} S\}$$

and at least one of the cardinals $\dim S, \operatorname{codim} S$ coincides with $\aleph$. For any cardinal $\alpha \le \aleph$ we define

$$\mathcal{G}_\alpha := \{S \in \mathcal{S} \mid \dim S = \alpha, \quad \operatorname{codim} S = \aleph\},$$
$$\mathcal{G}^\alpha := \{U \in \mathcal{S} \mid \dim S = \aleph, \quad \operatorname{codim} S = \alpha\}$$

(recall that $\mathcal{S}$ is the set of proper subspaces of V). Then $\mathcal{G}_\aleph = \mathcal{G}^\aleph$.

Let $\mathcal{P}$ be the projective space associated with V and let B be a base of $\mathcal{P}$. The set $\mathcal{B}_\alpha$ $(\mathcal{B}^\alpha)$ consisting of all elements of $\mathcal{G}_\alpha$ $(\mathcal{G}^\alpha)$ spanned by points of B is said to be the *base subset* defined by the base B. It is clear that $\mathcal{B}_1 = B$ and $\mathcal{B}^1$ is an independent subset of the dual projective space $\mathcal{P}^*$, but $\mathcal{B}^1$ is not a base of $\mathcal{P}^*$ (the dimension of $\mathcal{P}^*$ is greater than $\aleph$, see [2]).

Denote by $\mathfrak{B}_\alpha$ and $\mathfrak{B}^\alpha$ the families of all base subsets of $\mathcal{G}_\alpha$ and $\mathcal{G}^\alpha$, respectively. Any semilinear automorphism of V induces automorphisms of the *Grassmann block spaces* $\mathrm{GB}_\alpha = (\mathcal{G}_\alpha, \mathfrak{B}_\alpha)$ and $\mathrm{GB}^\alpha = (\mathcal{G}^\alpha, \mathfrak{B}^\alpha)$.

Theorem. *If $\alpha < \aleph$ then any automorphism of* GB_α *is induced by a semilinear automorphism of V.*[1]

For GB_1 this is the Fundamental Theorem of Projective Geometry (three points of $\mathcal{P}$ are non-collinear if and only if there is a base of $\mathcal{P}$ containing them, this implies that an automorphism of GB_1 is a collineation of $\mathcal{P}$).

3. Base subsets

3.1. Inexact subsets

Let $\mathcal{G}$ be a Grassmannian of V. Let also $B = \{P_a\}_{a \in \Delta}$ be a base of $\mathcal{P}$ and $\mathcal{B}$ be the base subset of $\mathcal{G}$ defined by B. We write $\mathcal{B}(+a)$ and $\mathcal{B}(-a)$ for the set of all elements of $\mathcal{B}$ which contain P_a or do not contain P_a, respectively. Such sets will be called *simple subsets* of $\mathcal{B}$. We also define

$$\mathcal{B}(+a, +b) := \mathcal{B}(+a) \cap \mathcal{B}(+b) \qquad \text{and} \qquad \mathcal{B}(+a, -b) := \mathcal{B}(+a) \cap \mathcal{B}(-b).$$

Let $\mathcal{X}$ be a subset of $\mathcal{B}$ and $\mathcal{X}_a$ be the set of all elements of $\mathcal{X}$ containing P_a. If $\mathcal{X}_a$ is not empty then we write $S_a(\mathcal{X})$ for the intersection of all elements of $\mathcal{X}_a$. If $\mathcal{X}_a$ is empty then $S_a(\mathcal{X}) := \emptyset$. We say that $\mathcal{X}$ is an *exact* subset of $\mathcal{B}$ if there is only one base subset of $\mathcal{G}$ containing $\mathcal{X}$; otherwise, $\mathcal{X}$ is said to be *inexact*. In other words, $\mathcal{X}$ is exact if and only if $S_a(\mathcal{X}) = P_a$ for every $a \in \Delta$.

Example 1. Suppose that

$$\mathcal{X} = \mathcal{B}(+a, +b) \cup \mathcal{B}(-a) \tag{1}$$

and $a \neq b$. Then $S_d(\mathcal{X}) = P_d$ if $d \neq a$ and $S_a(\mathcal{X}) = P_a + P_b$. So $\mathcal{X}$ is inexact. Any $U \in \mathcal{B} \setminus \mathcal{X}$ intersects $P_a + P_b$ by P_a, thus $S_a(\mathcal{X} \cup \{U\}) = P_a$ and $\mathcal{X} \cup \{U\}$ is exact. This means that the inexact subset $\mathcal{X}$ is maximal.

Lemma 1. *If $\mathcal{X}$ is a maximal inexact subset of $\mathcal{B}$ then* (1) *holds for some distinct $a, b \in \Delta$.*

Proof. Since $\mathcal{X}$ is inexact, we have $S_a(\mathcal{X}) \neq P_a$ for some $a \in \Delta$. If $S_a(\mathcal{X})$ is not empty then we choose $b \in \Delta \setminus \{a\}$ such that P_b is contained in $S_a(\mathcal{X})$. If $S_a(\mathcal{X}) = \emptyset$ then we can take any $b \in \Delta \setminus \{a\}$. For each of these cases $\mathcal{X}$ is contained in the maximal inexact subset $\mathcal{B}(+a, +b) \cup \mathcal{B}(-a)$. Since $\mathcal{X}$ is a maximal inexact subset, we get (1). $\square$

We say that $\mathcal{X} \subset \mathcal{B}$ is a *complement subset* of $\mathcal{B}$ if $\mathcal{B} \setminus \mathcal{X}$ is a maximal inexact subset. For this case

$$\mathcal{X} = \mathcal{B} \setminus (\mathcal{B}(+a, +b) \cup \mathcal{B}(-a)) = \mathcal{B}(+a, -b)$$

for some distinct $a, b \in \Delta$; in other words, $\mathcal{X}$ is the intersection of two simple subsets of $\mathcal{B}$ having different types.

For two distinct complement subsets $\mathcal{B}(+a, -b)$ and $\mathcal{B}(+a', -b')$ one of the following possibilities is realized:

[1] The reason for restricting ourself to this partial case will be given in Section 5.

(1) $a = a'$ or $b = b'$,
(2) $a = b'$ or $b = a'$, then the intersection of $\mathcal{B}(+a, -b)$ and $\mathcal{B}(+a', -b')$ is empty,
(3) $\{a, b\} \cap \{a', b'\} = \emptyset$.

In the first case our complement subsets are said to be *adjacent*.

Lemma 2. *For two distinct complement subsets $\mathcal{X}, \mathcal{Y} \subset \mathcal{B}$ there exist adjacent complement subsets $\mathcal{X}', \mathcal{Y}' \subset \mathcal{B}$ such that $\mathcal{X} \cap \mathcal{Y} \subset \mathcal{X}' \cap \mathcal{Y}'$ and the inverse inclusion holds only for the case when $\mathcal{X}$ and $\mathcal{Y}$ are adjacent.*

Proof. Easy verification. $\square$

By Lemma 2, the intersection of two distinct complement subsets of $\mathcal{B}$ is maximal if and only if they are adjacent.

A collection of complement subsets of $\mathcal{B}$ will be called *A-collection* if any two distinct elements are adjacent. For each $a \in \Delta$

$$\{\mathcal{B}(+a, -b)\}_{b \in \Delta \setminus \{a\}} \qquad \text{and} \qquad \{\mathcal{B}(+b, -a)\}_{b \in \Delta \setminus \{a\}}$$

are maximal A-collections. It is not difficult to prove that every maximal A-collection is of one of these types. Thus each simple subset of $\mathcal{B}$ can be characterized as the union of all elements of a certain maximal A-collection. The simple subsets $\mathcal{B}(+a)$ and $\mathcal{B}(-a)$ are said to be of *the first type* and *the second type*, respectively.

3.2. Special transformations

Let $\mathcal{B}$ be as in the previous subsection and $\mathcal{B}'$ be the base subset of $\mathcal{G}$ defined by the other base $B' = \{P'_a\}_{a \in \Delta}$ of $\mathcal{P}$. We write $\mathcal{B}'(+a)$ and $\mathcal{B}'(-a)$ for the set of all elements of $\mathcal{B}'$ which contain P'_a or do not contain P'_a, respectively.

A bijection $g : \mathcal{B} \to \mathcal{B}'$ will be called *special* if g and g^{-1} map inexact subsets to inexact subsets.

Lemma 3. *Let $g : \mathcal{B} \to \mathcal{B}'$ be a special bijection. Then g preserves the class of simple subsets, and there exists a bijective transformation $\delta : \Delta \to \Delta$ such that*

$$g(\mathcal{B}(+a)) = \mathcal{B}'(+\delta(a)), \qquad g(\mathcal{B}(-a)) = \mathcal{B}'(-\delta(a)) \qquad \forall\, a \in \Delta$$

or

$$g(\mathcal{B}(+a)) = \mathcal{B}'(-\delta(a)), \qquad g(\mathcal{B}(-a)) = \mathcal{B}'(+\delta(a)) \qquad \forall\, a \in \Delta.$$

Proof. Clearly, g preserves the class of maximal inexact subsets. So g and g^{-1} map complement subsets to complement subsets. By Lemma 2, the adjacency relation for complement subsets is preserved and maximal A-collections go to maximal A-collections in both directions. Thus the class of simple subsets is invariant. Two distinct simple subsets of $\mathcal{B}$ have different types if and only if their intersection is empty or a complement subset. This implies that g preserves the types of all simple subsets or changes the type of each of them. Since $\mathcal{B}(-a) = \mathcal{B} \setminus \mathcal{B}(+a)$ for any $a \in \Delta$, there exists a bijective transformation $\delta : \Delta \to \Delta$ satisfying the required conditions. $\square$

We say that a special bijection is of *the first type* if it preserves the types of all simple subsets; otherwise, it is said to be of *the second type*.

Let $S, U \in \mathcal{B}$. The equality $S \cap U = P_a$ implies that the dimension of S and U is not greater than the codimension, so it is possible only for the case when $\mathcal{G} = \mathcal{G}_\alpha$. This equality means that $S, U \in \mathcal{B}(+a)$ and S or U does not belong to $\mathcal{B}(+b)$ for every $b \in \Delta \setminus \{a\}$.

Similarly, we can get $S + U = \langle B \setminus \{P_a\}\rangle$ only for the case when the dimension of S and U is $\aleph$, in other words, $\mathcal{G} = \mathcal{G}^\alpha$. This equality holds if and only if $S, U \in \mathcal{B}(-a)$ and S or U does not belong to $\mathcal{B}(-b)$ for every $b \in \Delta \setminus \{a\}$.

Thus Lemma 3 gives the following.

Lemma 4. *Let g and δ be as in the previous lemma. Let also $S, U \in \mathcal{B}$. If g is of the first type and $\mathcal{G} = \mathcal{G}_\alpha$ then*

$$S \cap U = P_a \Longleftrightarrow g(S) \cap g(U) = P'_{\delta(a)}.$$

If g is of the first type and $\mathcal{G} = \mathcal{G}^\alpha$ then

$$S + U = \langle B \setminus \{P_a\}\rangle \Longleftrightarrow g(S) + g(U) = \langle B' \setminus \{P'_{\delta(a)}\}\rangle.$$

If g is of the second type then

$$S \cap U = P_a \Longleftrightarrow g(S) + g(U) = \langle B' \setminus \{P'_{\delta(a)}\}\rangle,$$
$$S + U = \langle B \setminus \{P_a\}\rangle \Longleftrightarrow g(S) \cap g(U) = P'_{\delta(a)},$$

and $\mathcal{G} = \mathcal{G}_\aleph$.

Thus special bijections of the second type may exist only for $\mathcal{G} = \mathcal{G}_\aleph$.

4. Proof of theorem

For any P belonging to $\mathcal{G}_1 \cup \mathcal{G}^1$ we denote by $\mathcal{G}(P)$ the set of all elements of $\mathcal{G}$ incident with P (two subspaces are incident if one of them is contained in the other). Such sets will be called *simple subsets* of $\mathcal{G}$. Any simple subset of a base subset can be extended to a unique simple subset of $\mathcal{G}$.

Lemma 5. *Let $\mathcal{B}$ be a base subset of $\mathcal{G} = \mathcal{G}_\alpha$ and let $B = \{P_a\}_{a\in\Delta}$ be the base of $\mathcal{P}$ associated with $\mathcal{B}$. For any $a \in \Delta$ and any $S \in \mathcal{G}(P_a) \setminus \mathcal{B}$ there exist $M, N \in \mathcal{B}(+a)$ such that $M \cap N = P_a$ and M, N, S are contained in a base subset of $\mathcal{G}$.*

To prove Lemma 5 we need the following fact connected to infinite cardinals.

Fact. *Let X be a set of an infinite cardinal α. Then X contains a family of non-intersecting subsets $\{Y_a\}_{a\in\Gamma}$, where $|Y_a| = \alpha$ for each $a \in \Gamma$ and $|\Gamma| = \alpha$. In particular, there exist two subsets Y, Z of X such that $|Y| = |Z| = \alpha$ and $|Y \cap Z| = 1$.*

Proof of the Fact. Since $|X^2| = \alpha$, we replace X by X^2 and consider the subsets $x \times X$, $x \in X$. □

Proof of Lemma 5. If $\alpha < \infty$ then we take a $(2\alpha - 1)$-dimensional subspace T spanned by elements of B and intersecting S by P_a. We choose $M, N \in \mathcal{B}(+a)$ such that $M + N = T$, as required.

Suppose that α is infinite. Then there exists a collection $\{U_b\}_{b\in\Gamma} \subset \mathcal{B}$ such that $U_b \cap U_c = 0$ if $b \neq c$ and $|\Gamma| = \alpha$. For any $b \in \Gamma$ we choose $P_b \in B \cap (U_b \setminus S)$ (this is possible, since

$S \neq U_b$). All P_b are different and $\sum_{b \in \Gamma} P_b \in \mathcal{B}$. Then

$$T := P_a + \sum_{b \in \Gamma} P_b \in \mathcal{B}(+a).$$

We take any $M, N \in \mathcal{B}(+a)$ contained in T and such that $M \cap N = P_a$. Since $S \cap T = P_a$, we can construct a base subset of $\mathcal{G}$ containing M, N, S. $\square$

Now suppose that $\mathcal{G} = \mathcal{G}_\alpha$, $\alpha < \aleph$ and consider an automorphism f of the Grassmann block space GB_α.

Let $P \in \mathcal{G}_1$. We take a base subset $\mathcal{B} \subset \mathcal{G}$ such that P belongs to the base of $\mathcal{P}$ associated with $\mathcal{B}$. Then $f|_{\mathcal{B}}$ is a special bijection to $f(\mathcal{B})$. Since $\alpha < \aleph$, it is of first type and we have

$$f(\mathcal{B} \cap \mathcal{G}(P)) \subset \mathcal{G}(T)$$

for a certain $T \in \mathcal{G}_1$. We want to show that $f(\mathcal{G}(P)) = \mathcal{G}(T)$.

Let us take any $S \in \mathcal{G}(P) \setminus \mathcal{B}$. By Lemma 5, we can choose $M, N \in \mathcal{B} \cap \mathcal{G}(P)$ such that $M \cap N = P$ and there exists a base subset $\mathcal{B}'$ containing M, N, S. Then P belongs to the base of $\mathcal{P}$ associated with $\mathcal{B}'$ and

$$f(\mathcal{B}' \cap \mathcal{G}(P)) \subset \mathcal{G}(T')$$

for some $T' \in \mathcal{G}_1$. Then

$$T = f(M) \cap f(N) = T'$$

(Lemma 4). We have established that $f(\mathcal{G}(P)) \subset \mathcal{G}(T)$. Since f^{-1} is an automorphism of GB_α, using arguments given above we can prove the inverse inclusion.

Thus f induces a bijective transformation $f' : \mathcal{G}_1 \to \mathcal{G}_1$. This bijection is an automorphism of GB_1.

Proof. Let B be a base of $\mathcal{P}$ and let $\mathcal{B}$ be the base subset of $\mathcal{G}$ defined by B. Let also B' be the base of $\mathcal{P}$ associated with $f(\mathcal{B})$. Then $f'(B) = B'$. Similarly, we establish that the inverse mapping transfers bases to bases. $\square$

Therefore, f' is induced by a semilinear automorphism of V (the Fundamental Theorem of Projective Geometry). This automorphism induces f.

5. Final remarks

In order to apply the results of Section 3 to $\mathcal{G}_\aleph$ and $\mathcal{G}^\alpha$ ($\alpha < \aleph$) we need the following statement "dual" to Lemma 5.

Let $\mathcal{B}$ be a base subset of $\mathcal{G} = \mathcal{G}^\alpha$ and let $B = \{P_a\}_{a \in \Delta}$ be the base of $\mathcal{P}$ associated with $\mathcal{B}$. For any $a \in \Delta$ and any $S \in \mathcal{G}(\langle B \setminus \{P_a\} \rangle) \setminus \mathcal{B}$ there exist $M, N \in \mathcal{B}(-a)$ such that $M + N = \langle B \setminus \{P_a\} \rangle$ and M, N, S are contained in a base subset of $\mathcal{G}$.

In contrast to the finite-dimensional case, we cannot use the duality principles; so it is impossible to derive this statement immediately from Lemma 5.

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